A Look At Markowitz Portfolio Theory

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Definitions

Let X_o be the amount of capital invested in an asset at time 0. Let X_T be the amount received when the asset is sold at time T. Then the *total return* is

$$R \equiv \frac{X_t}{X_o} \tag{1}$$

The *rate of return* is

$$r \equiv \frac{X_t - Xo}{X_o} = R - 1 \tag{2}$$

You are considered to be profitable if $r > 0$.

Problem Setup

A portfolio is a collection of assets. We want to derive a way to pick the best portfolio we can, given a set of assets. Assume you have n assets. Let R_i be the total return for the i^{th} asset for i=1,2,..,n (and similarly for r_i).

Let X_o be to the total amount of money invested in the portfolio. Let X_{oi} be the amount of capital invested into the i^{th} asset, and let $X_{oi} > 0$.

We also want to consider the relationship between each X_{oi} and X_o. We will use a fractional weight, that is for the i^{th} asset:

$$w_i \equiv \frac{X_{oi}}{X_o}, \tag{3}$$

$$\sum w_i = 1 \tag{4}$$

So now we have that the amount of capital invested into the i^{th} asset at time 0 is:

$$X_{oi} = w_i X_o \tag{5}$$

And the amount of capital generated at time T is $R_i w_i X_o$.

So the *total return of the portfolio at time T* is:

$$\begin{aligned} R &= \frac{\sum R_i w_i X_o}{X_o} && (6)\\ &= \frac{\sum (1 + r_i) w_i X_o}{X_o} && (7)\\ &= \sum w_i + \sum r_i w_i && (8)\\ &= 1 + \sum r_i w_i && (9) \end{aligned}$$

And now using (2) we get that for the portfolio:

$$r = \sum w_i r_i \tag{10}$$

Expected Value and Variance of the Portfolio

Suppose for each asset we know its mean rate of return ($\bar{r_i}$), its variance (σ_i^2), and the covariance between it and each other asset (σ_{ij}). We then can calculate the expected return of the portfolio:

$$\begin{aligned}
\bar{r} &= E[r] && (11)\\
&= \sum w_i E[r_i] && (12)\\
&= \sum w_i \bar{r_i} && (13)
\end{aligned}$$

The variance for the portfolio can also be calculated:

$$\begin{aligned}
\sigma^2 &= E[(r - \bar{r})] && (14)\\
&= E\left[(\sum w_i r_i - \sum w_i \bar{r_i})^2\right] && (15)\\
&= E\left[\left(\sum_i w_i(r_i - \bar{r_i})\right)\left(\sum_j w_j(r_j - \bar{r_j})\right)\right] && (16)\\
&= E\left[\sum_i \sum_j w_i w_j (r_i - \bar{r_i})(r_j - \bar{r_j})\right] && (17)\\
&= \sum_i \sum_j w_i w_j \sigma_{ij} && (18)
\end{aligned}$$

The Markowitz Problem

An ideal portfolio would be one that insures great profits with complete certainty. This doesn't happen in the real world, but we want to come as close to it as possible. This can be worded in two ways. We either want to minimize risk for a fixed return:

$$\min \sum_i \sum_j w_i w_j \sigma_{ij}$$

subject to

$$\bar{r} = \sum_i w_i \bar{r_i}$$

$$\sum_i w_i = 1$$

Or you can also say you want to maximize profit for a fixed risk:

$$\max \sum_i w_i \bar{r_i}$$

subject to

$$\sigma^2 = \sum_i \sum_j w_i w_j \sigma_{ij}$$

$$\sum_i w_i = 1$$

I am only going to work with the first problem, since working with the second is pretty much equivalent. I am going to use Lagrange Multipliers in order to solve the minimization problem. I will not introduce the theory of Lagrange Multipliers in depth since it is something taught in a previous class. In short, you can use Lagrange multipliers to find the extrema of a multivariate function f which is subject to multivariate function constraints (in this case 2) g_1, g_2. We are looking for areas that the gradients of the functions line up (that is, become a multiple of each other). So in summary for an extrema of f to exist on g it is true that the gradients are a multiple of each other. This multiple we will call λ.

So now to rephrase our minimization problem into a Lagrange notation we have:

$$f(w_1, .., w_n) = \sum_i \sum_j w_i w_j \sigma_{ij}$$

$$g_1(w_1, .., w_n) = (\sum_i w_i \bar{r_i}) - \bar{r}$$

$$g_2(w_1, .., w_n) = (\sum_i w_i) - 1$$

We form the Lagrangian by summing up the functions, using multipliers λ_1 and λ_2:

$$L(w_1, .., w_n) = f + \lambda_1 g_1 + \lambda_2 g_2 = 0 \tag{19}$$

As a reminder, we know the values of $\bar{r}, \bar{r_i}$, and σ_{ij}. We are trying to find the values of $w_1, .., w_n, \lambda_1, \lambda_2$.

We now have one equation with n+2 unknowns, so we need to come up with some more conditions. From the setup we said that the gradients of the functions are going to line up, that is:

$$\nabla f(x_1,..,x_n) = -\lambda \nabla g(x_1,..,x_n) \tag{20}$$

If this is true then it must also be true that each directional derivative are also multiples of each other:

$$\frac{\partial f}{\partial x_i} = -\lambda \frac{\partial g}{\partial x_i} \tag{21}$$

For i=1,2,..,n

We can apply this to our case. Referring back to (19) we have:

$$\frac{\partial L(w_1,..,w_n)}{\partial w_i} = 0, \text{ for } i = 1,2,..,n \tag{22}$$

Taking these derivatives will give us n more equations. We also know that:

$$\lambda_1 g_1 = \lambda_2 g_2 = 0 \tag{23}$$

Using (19),(22), and (23) we have a system of n+2 equations with n+2 unknowns. This system is solvable using linear algebra techniques, and will yield the optimal values of $w_1, w_2, .., w_n$.

Summary

In this analysis I have shown the mathematical derivation of creating an optimally low risk portfolio for a given expected return. This was done using probability and Lagrange multipliers. To summarize the steps you would take to create one of these portfolios:

1. Choose the n assets you wish to include in your portfolio
2. Calculate the expected return of each asset, its variance, and its covariance to each of the other assets.
3. Choose the expected return you want to get from the portfolio
4. Solve the system of linear equations given by (19),(22), and (23) for $w_1, w_2, .., w_n$.
5. For each asset i, invest $w_i X_o$ into it